Methodology for Deriving Eigenvalue Formulas in Steering and Body Systems under Force-Controlled Conditions

Hideki Sakai*1

The natural frequency and damping ratio of the steering system are essential metrics for describing its dynamic characteristics. Since the driver rotates the steering wheel by applying torque, these metrics are critical for designing a more direct steering feel. They are obtained by solving a fourth-order characteristic equation. To achieve a compact representation, a three-step procedure has previously been proposed to derive approximate solutions. In this study, an alternative formulation is developed based on the known property that the natural frequencies are independent of vehicle speed. This property allows the derivation of eigenvalue formulas through purely algebraic operations. The resulting process is expected to facilitate clearer understanding and enable more precise application of the metrics in designing the dynamic behavior of steering systems.

Keywords: Steering-system Dynamics, Eigen Values, Natural Frequencies, Force control, Automobile

1. Introduction

The natural frequency and damping ratio of a steering system serve as key metrics for evaluating its dynamic characteristics. Because the driver steers by applying torque to the steering wheel, these metrics are essential for designing a more direct and responsive handling feel. These metrics are derived by solving a fourth-order characteristic equation. To obtain a concise representation, a three-step procedure has previously been used to derive an approximate solution[1].

In order to enhance the clarity of the derivation, this paper presents a method that utilizes the fact that the natural frequencies are independent of vehicle speed. Based on this property, an algebraic procedure is introduced to derive a closed-form expression for the eigenvalues. This clear process is expected to enable a more effective use of the metrics in the design of the steering system's dynamic response.

2. Characteristic Equation

The motion of the steering system is described by Newton's second law[1]. The torque applied by the driver to the steering system, denoted as T_h, is shown in Fig. 1. Another torque acting on the system is the product of the trail ξ, acting as a moment arm, and the front cornering force $2F_f$. The moment of inertia of the steering system is denoted as I_s, and its typical value is approximately 15 kg·m². These quantities yield the rotational form of Newton's second law:

$$I_S\ddot{\delta} = -2F_f\xi + T_h \tag{1}$$

The cornering forces are obtained from the planar motion model of the vehicle shown in Fig. 2. The front and rear cornering forces are expressed as:

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*1 Associate Professor, Department of Robotics, Faculty of Engineering, Kindai University; Associate Professor, Fundamental Technology for Next Generation Research Institute, Kindai University
1 Takaya Umenobe, Higashi-Hiroshima City, Hiroshima 739-2116, Japan
E-mail sakai@hiro.kindai.ac.jp
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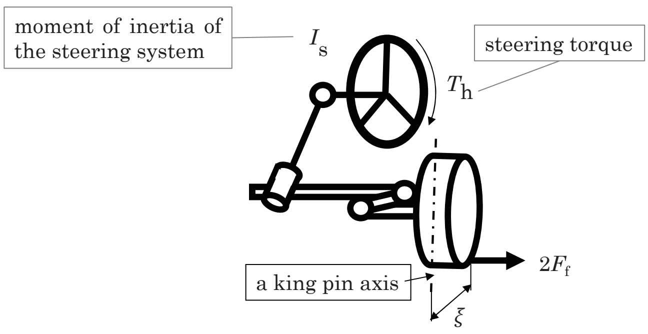

Fig. 1 Model of steering system

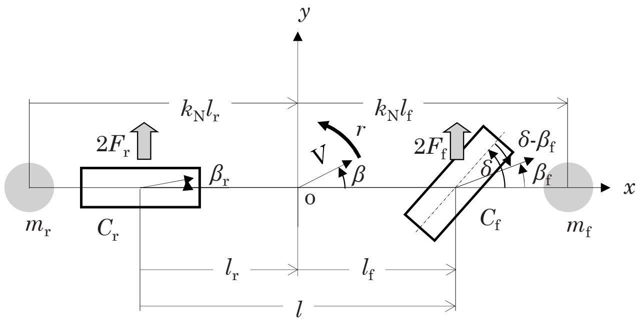

Fig. 2 Vehicle planar model

$$2F_{\mathrm{f}} = -C_{\mathrm{f}} m_{\mathrm{f}} (\beta_{\mathrm{f}} - \delta) \tag{2}$$

$$2F_{\mathrm{r}} = -C_{\mathrm{r}} m_{\mathrm{r}} \beta_{\mathrm{r}} \tag{3}$$

where C_{f} and C_{r} are known as cornering stiffness coefficients[1], with typical values of C_{f}=100 and C_{r}=200. The masses supported by the front and rear axles, denoted as m_{f} and m_{r} respectively, satisfy:

$$m_{\mathrm{f}} = \frac{l_{\mathrm{r}}}{l} m \tag{4}$$

$$m_{\mathrm{r}} = \frac{l_{\mathrm{f}}}{l} m \tag{5}$$

Here, l_{f} and l_{r} are the distances from the center of gravity to the front and rear axles, both typically around 1.25 m, and $l = l_{\mathrm{f}} + l_{\mathrm{r}}$ is the wheelbase. The sideslip angles β_{f} and β_{r} at the front and rear axles can be expressed in terms of the sideslip angle at the center of mass β, yaw rate r, and vehicle speed V as follows:

$$\beta_{\mathrm{f}} = \beta + \frac{l_{\mathrm{f}}}{V} r \tag{6}$$

$$\beta_{\mathrm{r}} = \beta - \frac{l_{\mathrm{r}}}{V} r \tag{7}$$

Using these relationships, Newton's law for linear motion provides the lateral force balance. The lateral acceleration is expressed as $V(\mathrm{r} + \dot{\beta})$, which leads to

$$mV(\mathrm{r} + \dot{\beta}) = 2F_{\mathrm{f}} + 2F_{\mathrm{r}} \ . \tag{8}$$

Newton's law for rotational motion around the center of gravity uses the yaw moment of inertia I_z[1]:

$$I_z = k_{\mathrm{N}}{}^2 m_{\mathrm{f}} l_{\mathrm{f}}{}^2 + k_{\mathrm{N}}{}^2 m_{\mathrm{r}} l_{\mathrm{r}}{}^2 = k_{\mathrm{N}}{}^2 m l_{\mathrm{f}} l_{\mathrm{r}} \tag{9}$$

This inertia is often represented by the dimensionless "dynamic index[2]" k_{N}^2, where k_{N} is the yaw radius ratio[1]. Typical values for k_{N}^2 and k_{N} are 1. Using Eq. (9) gives Newton's law for rotational motion:

$$k_{\mathrm{N}}{}^2 m l_{\mathrm{f}} l_{\mathrm{r}} \dot{r} = 2F_{\mathrm{f}} l_{\mathrm{f}} - 2F_{\mathrm{r}} l_{\mathrm{r}} \tag{10}$$

Equations (1), (8), and (10) can be combined to yield the fourth-order characteristic equation:

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \tag{11}$$

$$a_3 \cong \frac{C_{\mathrm{f}} + C_{\mathrm{r}}}{k_{\mathrm{N}} V} \tag{12}$$

$$a_2 = \frac{C_{\mathrm{r}}}{k_{\mathrm{N}}{}^2 l} + \left(\frac{C_{\mathrm{r}} l}{V^2} - 1\right) \frac{C_{\mathrm{f}}}{k_{\mathrm{N}}{}^2 l} + \frac{C_{\mathrm{f}} m_{\mathrm{f}} \xi}{I_{\mathrm{S}}} \tag{13}$$

$$a_1 \cong \frac{C_{\mathrm{f}} m_{\mathrm{f}} \xi}{I_{\mathrm{S}}} \frac{C_{\mathrm{r}}}{k_{\mathrm{N}} V} \tag{14}$$

$$a_0 \cong \frac{C_{\mathrm{f}} m_{\mathrm{f}} \xi}{I_{\mathrm{S}}} \frac{C_{\mathrm{r}}}{k_{\mathrm{N}}{}^2 l} \tag{15}$$

Equations (12) and (14) are approximated using the relations $l_{\mathrm{f}} \cong l_{\mathrm{r}}$, $k_{\mathrm{N}}{}^2 \cong 1$ and $k_{\mathrm{N}} \cong 1$.

The factorability of Eq. (11) depends on certain conditions[3]. These conditions are governed by the dimensionless steering system inertia[4], I_{sN}, defined as:

$$I_{\mathrm{SN}} \equiv \frac{I_{\mathrm{S}}}{k_{\mathrm{N}}{}^2 m_{\mathrm{f}} \xi l} \tag{16}$$

When I_{sN} >1/6, the system may exhibit instability[5]. In such cases, ensuring system stability becomes the primary design objective, and the eigenvalues of the system are of secondary importance. Conversely, when I_{sN} <1/6, instability does not occur[5], and the responsiveness (determined by the eigenvalues) becomes the principal concern. This paper focuses solely on the case where I_{sN} <1/6.

3. Conventional Procedure for Eigenvalue Formulation

This section outlines the previously proposed three-step approach, which yields compact eigenvalue formulas with practical accuracy despite the complexity of the original fourth-order equation.

3.1. Exact Characteristic Equation under Simplified Conditions

Substituting $C_f = C_r = C$ and k_N=1 into Eqs. (11)–(15) allows the equation to be factorized exactly[6]:

$$\left[s^2+\frac{C}{V}s+\left(1+\sqrt{1-4I_{SN}}\right)\frac{Cm_f\xi}{I_S}\right]\cdot\left[s^2+\frac{C}{V}s+\left(1-\sqrt{1-4I_{SN}}\right)\frac{Cm_f\xi}{I_S}\right]=0 \quad (17)$$

Assuming $4I_{SN} \ll 1$, applying the Maclaurin expansion with respect to I_{sN} yields:

$$\left[s^2+\frac{C}{V}s+c\frac{Cm_f\xi}{I_S}\right]\cdot\left[s^2+\frac{C}{V}s+\frac{C}{l}\right]=0 \quad (18)$$

3.2. Hypothetical Characteristic Equation under General Conditions

This result suggests a hypothetical characteristic equation for the general case where $C_f \neq C_r$[1]. The first bracket in the factored form includes the design parameters of the steering system, such as I_s, ξ, and I_{sN}, while the second bracket reflects vehicle-side parameters. Therefore, the characteristic equation can be hypothetically written as:

$$\left[s^2+\frac{C_f}{V}s+(1-I_{SN})\frac{C_f m_f\xi}{I_S}\right]\cdot\left[s^2+\frac{C_r}{V}s+\frac{C_r}{l}\right]=0 \quad (19)$$

To generalize the formulation beyond k_N=1, the coefficient of s^1 in the first bracket is assumed by referencing the case without steering system dynamics, which gives a term of $(C_f + C_r)/(k_N V)$. Thus, it is reasonable to define[1]:

$$\left[s^2+\frac{C_f}{k_N V}s+\frac{C_f m_f\xi}{I_S}\right]\cdot\left[s^2+\frac{C_r}{k_N V}s+\frac{C_r}{k_N{}^2 l}\right]=0 \quad (20)$$

To maintain consistency in format between the two brackets, the explicit appearance of I_{sN} inside the round brackets is eliminated.

At this stage, the errors in the absolute value of the eigen-frequency compared with the numerical solution remain below 10%, while the error in fast responses (real parts) are below 1%.

3.3. Correction by Coefficient Matching

A more accurate formula can be obtained by introducing a correction term to Eq. (19)[1]. Adding small correction terms $\Delta \ll 1$ to Eq. (19) gives:

$$\left[(1+\Delta_{s2})s^2+(1+\Delta_{s1})\frac{C_f}{k_N V}s+\frac{C_f m_f\xi}{I_S}\right]\cdot\left[(1+\Delta_{b2})s^2+(1+\Delta_{b1})\frac{C_r}{k_N V}s+\frac{C_r}{k_N{}^2 l}\right]=0 \quad (21)$$

Expanding this equation and eliminating products of correction terms, then comparing the result with Eqs. (11)–(15), allows the correction terms to be determined as:

$$\Delta_{s2}=-\Delta_{b2}=I_{SN} \quad (22)$$

$$\Delta_{s1}=-\Delta_{b1}=I_{SN} \quad (23)$$

Therefore, the eigenvalues are defined as follows:

$$\omega_s \approx \sqrt{1-I_{SN}}\sqrt{\frac{C_f m_f\xi}{I_S}} \quad (24)$$

$$\approx \sqrt{\frac{C_f m_f\xi}{I_S}} \quad (25)$$

$$\varsigma_s\omega_s \cong\cdot \frac{C_f}{2k_N V} \quad (26)$$

$$\omega_b \approx \sqrt{1+I_{SN}}\sqrt{\frac{C_r}{k_N{}^2 l}} \tag{27}$$

$$\approx \sqrt{\frac{C_r}{k_N{}^2 l}} \tag{28}$$

$$\varsigma_b\omega_b \cong \cdot \frac{C_r}{2k_N V} \tag{29}$$

Here, ω_s is the natural frequency of the steering system, $\zeta_s\omega_s$ represents its fast response, ω_b is the natural frequency of the vehicle system excluding the steering system, and $\zeta_b\omega_b$ is the fast response of the vehicle system.

Comparing these frequencies with numerical simulation results shows a maximum error of only 1%[1]. Thus, the derived formulas are compact yet sufficiently accurate for practical design. However, since the derivation relies on intuitive steps, some engineers may find the justification insufficiently rigorous.

4. New Procedure for Eigenvalue Formulation

While the exact solution to the characteristic equation becomes overly complex, practical applications often require approximate solutions. Such approximations inherently involve assumptions. This chapter presents a new derivation procedure that relies solely on a widely recognized assumption and uses algebraic operations to derive the eigenvalue formulas deductively.

4.1. Assumption

This study, therefore, assumes that both ω_s and ω_b are independent of vehicle speed. Numerical studies have shown that the natural frequencies of both the steering system and the vehicle system remain nearly constant regardless of vehicle speed[7]. This observation is consistent with the absence of V in Eqs. (24) and (27). This assumption enables the use of any hypothetical vehicle speed in the derivation of the natural frequencies.

4.2. New Derivation of Natural Frequencies

The assumption in the previous section allows the derivation of ω_s and ω_b under the hypothetical condition of infinite vehicle speed. Substituting Eq. (16) and $V=\infty$ into Eq. (11) allows us to describe:

$$(s^2)^2 + \left(\frac{C_r}{k_N{}^2 l} - \frac{C_f}{k_N{}^2 l} + \frac{C_f}{I_{SN}k_N{}^2 l}\right)s^2 + \frac{C_f}{I_{SN}k_N{}^2 l}\cdot\frac{C_r}{k_N{}^2 l} = 0 \tag{30}$$

Substituting $s^2=-\omega^2$ into Eq. (30) and solving for ω^2 yields

$$\omega^2 = \frac{1}{2}\left\{1 \pm \sqrt{1 - \frac{\left(\frac{C_f C_r}{I_{SN}k_N{}^2 l^2 l}\right)}{\left[\frac{1}{2}\left(\frac{C_r}{k_N{}^2 l} - \frac{C_f}{k_N{}^2 l} + \frac{C_f}{I_{SN}k_N{}^2 l}\right)\right]^2}}\right\} \cdot \left(\frac{C_r}{k_N{}^2 l} - \frac{C_f}{k_N{}^2 l} + \frac{C_f}{I_{SN}k_N{}^2 l}\right). \tag{31}$$

Given the assumption $I_{sN}<1/6$ introduced in Chapter 2, applying the Maclaurin expansion with respect to I_{sN} yields

$$\omega_s \approx \sqrt{\left[1 - \frac{I_{SN}}{k_N{}^2}\left(1 - \frac{C_r}{C_f}\frac{I_{SN}}{k_N{}^2}\right)\right]}\sqrt{\frac{C_f m_f \xi}{I_S}} \tag{32}$$

$$\omega_b \approx \sqrt{\left[\left(1 + \frac{I_{SN}}{k_N{}^2}\right)\right]}\sqrt{\frac{C_r}{k_N{}^2 l}}. \tag{33}$$

These natural frequencies are nearly identical to those obtained in the previous chapter. Since the typical value of k_N is 1 and the condition $I_{sN}<1/6$ holds, Eq. (32) is approximately equal to Eq. (24), and similarly, Eq. (33) is approximately equal to Eq. (27).

These formulas can also be obtained by directly substituting $I_{sN}=0$ under the assumption $I_{sN}<1/6$.

$$\omega_s \approx \sqrt{\left[1 - \frac{I_{SN}}{k_N{}^2}\right]}\sqrt{\frac{C_f m_f \xi}{I_S}} \tag{34}$$

$$\approx \sqrt{[1 - I_{SN}]}\sqrt{\frac{C_f m_f \xi}{I_S}} \qquad \text{(Repeated)} \quad (24)$$

$$\approx \sqrt{\frac{C_f m_f \xi}{I_S}} \qquad \text{(Repeated)} \quad (25)$$

$$\omega_b \approx \sqrt{[(1 + I_{SN})]}\sqrt{\frac{C_r}{k_N{}^2 l}} \qquad \text{(Repeated)} \quad (27)$$

$$\approx \sqrt{\frac{C_r}{k_N{}^2 l}} \qquad \text{(Repeated)} \quad (28)$$

4.3. New Derivation of Responsiveness

The characteristic equations of the steering system and the vehicle system, treated independently, can be written as:

$$s^2 + 2\varsigma_s\omega_s s + {\omega_s}^2 = 0 \tag{35}$$

$$s^2 + 2\varsigma_b\omega_b s + {\omega_b}^2 = 0 \tag{36}$$

Multiplying Eqs. (35) and (36) yields:

$$(s^2 + 2\varsigma_s\omega_s s + {\omega_s}^2)(s^2 + 2\varsigma_b\omega_b s + {\omega_b}^2) = 0 \tag{37}$$

Expanding Eq. (37) and subtracting Eq. (11) gives the residual:

$$D_3 s^3 + D_2 s^2 + D_1 s + D_0 = 0 \tag{38}$$

Where

$$D_3 = 2\varsigma_s\omega_s + 2\varsigma_b\omega_b - \frac{\frac{l_f + {k_N}^2 l_r}{l}C_f + \frac{{k_N}^2 l_f + l_r}{l}C_r}{{k_N}^2 V} \tag{39}$$

$$D_2 = 4\varsigma_s\omega_s\varsigma_b\omega_b - \frac{C_f C_r}{{k_N}^2 V^2} \tag{40}$$

$$D_1 = 2\frac{(I_{SN} + {k_N}^2)C_r}{{k_N}^4 l}\varsigma_s\omega_s - 2\frac{{k_N}^2(I_{SN} - {k_N}^2)C_f + {I_{SN}}^2 C_r}{I_{SN}{k_N}^4 l}\varsigma_b\omega_b - \frac{({k_N}^2 l_f + l_r)C_f C_r}{I_{SN}{k_N}^2 l^2 V} \tag{41}$$

$$D_0 = 2\frac{(C_f + C_r)C_r}{{k_N}^6 l^2} I_{SN} - \frac{{C_r}^2}{{k_N}^8 l^2}{I_{SN}}^2 \tag{42}$$

For Eq. (37) to serve as a high-accuracy approximation, it is necessary that $D_3 \approx 0$, $D_2 \approx 0$, $D_1 \approx 0$, and $D_0 \approx 0$. In particular, $2\zeta_s\omega_s$ and $2\zeta_b\omega_b$ must be determined so that D_3, D_2, and D_1 vanish. Note that D_0 does not include either $2\zeta_s\omega_s$ or $2\zeta_b\omega_b$.

Solving the pair of equations $D_3=0$ and $D_2=0$ for $2\zeta_s\omega_s$ and $2\zeta_b\omega_b$ gives:

$$\varsigma_s\omega_s = \frac{\frac{l_f + {k_N}^2 l_r}{l}C_f + \frac{{k_N}^2 l_f + l_r}{l}C_r}{4{k_N}^2 V} + \frac{\sqrt{\left(\frac{l_f + {k_N}^2 l_r}{l}C_f + \frac{{k_N}^2 l_f + l_r}{l}C_r\right)^2 - 4{k_N}^2 C_f C_r}}{4{k_N}^2 V} \tag{43}$$

$$\varsigma_b\omega_b = \frac{\frac{l_f + {k_N}^2 l_r}{l}C_f + \frac{{k_N}^2 l_f + l_r}{l}C_r}{4{k_N}^2 V} - \frac{\sqrt{\left(\frac{l_f + {k_N}^2 l_r}{l}C_f + \frac{{k_N}^2 l_f + l_r}{l}C_r\right)^2 - 4{k_N}^2 C_f C_r}}{4{k_N}^2 V} \tag{44}$$

For simplicity, performing a Taylor expansion around "dynamic index" ${k_N}^2=1$ and substituting $l_f=l_r=$l/2l yields

$$\varsigma_s\omega_s \cong \frac{1 + {k_N}^2}{2}\cdot\frac{C_f}{2{k_N}^2 V}. \tag{45}$$

Further, performing a Taylor expansion around "yaw radius coefficient" $k_N=1$ obtains

$$\varsigma_s\omega_s \cong\cdot \frac{C_f}{2k_N V}. \quad \text{(Repeated)} \tag{26}$$

Similarly

$$\varsigma_b\omega_b \cong\cdot \frac{C_r}{2k_N V}. \quad \text{(Repeated)} \tag{29}$$

Solving other combinations such as $D_2=0$ and $D_1=0$, or $D_1=0$ and $D_3=0$, leads to the same results. Therefore, the responsiveness can also be expressed as Eqs. (26) and (29), confirming the consistency of the new formulation.

5. Discussion

The eigenvalue formulas presented in this study are derived deductively based on a single known empirical property, while all other steps rely on mathematical procedures. Since any approximate solution to the characteristic equation inevitably requires some assumptions, this study explicitly adopts one well-established and physically consistent assumption: the natural frequencies are independent of vehicle speed. This assumption has been supported by previous research and numerical validation.

Understanding the logical steps in the derivation helps clarify the conditions under which the formulas can be applied. As a result, engineers can use these formulas with confidence in designing the dynamic response of steering systems. In particular, interpreting the real parts of the eigenvalues as indicators of fast response enables the practical application of the formulas to achieve more responsive and precisely tuned steering behavior.

6. Conclusion

This study has proposed an intuitive and logically consistent method for deriving eigenvalue formulas for steering system dynamics. The derivation relies on only one known empirical assumption—namely, that the natural frequencies are independent of vehicle speed—while all other steps follow deductive mathematical procedures. Because of the clarity of this process, the resulting formulas are not only compact and accurate but also allow engineers to understand their range of applicability.

The formulas derived in this study are expected to contribute to the design of more responsive steering systems. Most of the references cited in this paper are authored by the present author. This is due to the fact that research in this particular field remains limited. The content of this paper is intended to be appropriately referenced in the forthcoming English-language book, tentatively titled Pragmatic Vehicle Dynamics: Designing Engaging Handling, to be published by an international publisher.

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